1*. Let $|f_n| \leq g_n$ a.e. on E. Show that $\exists A \subseteq E$ of measure zero such that

$$|f_n| \leq g_n \text{ everywhere on } E \setminus A, \ \forall n \in \mathbb{N}.$$

2*. Show the following version of Littlewood's 2nd Principle: Let $E \in \mathfrak{M}$ and $f \in \mathfrak{M}\mathcal{F}^{+}(E)$, a non-negative measurable function on E. Then $\exists$ a sequence

$$(\varphi_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}(E) \text{ s.t. } 0 \leq \varphi_n \uparrow f \text{ on } E$$

(a seq of non-negative simple functions, pointwisely increasing to f). Can φ_n be further required to vanish outside $[-n, n]$?

Hint: For each n, note that

$$[0, +\infty] = [0, n) \cup_{\circ} [n, +\infty]$$
$$= \bigcup_{i=1}^{n \cdot 2^n}{}_{\circ} \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) \cup_{\circ} [n, +\infty]$$

and so

$$E = \bigcup_{i=1}^{n\cdot 2^n}{}_{0}\ f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) \cup_0 f^{-1}\left([n,+\infty]\right)$$

3.* Let F be a closed subset of $\mathbb{R}$ and $\mathbb{R}\setminus F$ be expressed as the disjoint union of open intervals $I_1, I_2, \cdots$. Show that

(a) $(\overline{I_i} \setminus I_i) \cap \mathbb{R} \subseteq F$, $\forall i$

(b) If I is an open interval contained in $\mathbb{R}\setminus F$ intersecting some I_i then $I \subseteq I_i$.

(f is not nece. measurable)

4.* Let $f \in \mathcal{BF}(E)$, $m(E) < +\infty$. Let

$$A := \left\{ \int_E \varphi : \varphi \leq f \text{ on } E,\ \varphi \in \mathcal{S}(E) \right\}$$

$$B := \left\{ \int_E g : g \leq f \text{ on } E,\ g \in \mathcal{BMF}(E) \right\}.$$

Show that $\sup A = \sup B$, i.e.

$$\underline{\int}_E f = \sup B.$$

5*. Let $f \in \mathcal{B}\mathcal{F}(E)$, $g \in \mathcal{B}\mathcal{M}\mathcal{F}(E)$, $m(E) < +\infty$.

Show that

$$\underline{\int}_E (f+g) = \underline{\int}_E f + \int_E g .$$